Approximation of the surface impedance for a stratified medium

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APPROXIMATION DE L'IMPEDANCE DE SURFACE D'UN MILIEU STRATIFIE

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Abstract:

We consider the problem of calculating the pressure in a fluid domain overlying a stratified elastic medium. The idea is that for such calculations it suffices to know the transmission operator or surface impedance on the boundary between the fluid and elastic mediums instead of the parameters(density and Lamé coefficients) of the elastic medium. We define a class of approximate impedance operators in which one could try to identify the best approximate to the true impedance of the elastic medium in the sense that the corresponding pressure is the closest in terms of least squares to that associated with the real medium. Here we treat the direct problem associated with this inverse problem, not the inverse problem itself. We study the stability of the associated initial boundary value problem and its numerical resolution. Two numerical schemes are described, and a stability result is demonstrated. Numerical results are given. For the particular case of one single acoustic layer the approximate impedance corresponds to an absorbing boundary condition, so we obtain as special cases discretizations of absorbing boundary conditions of arbitrary orders.

Résumé:

Nous considérons le problème du calcul de la pression dans un milieu fluide surplombant un milieu élastique stratifié. L'idée est, que pour un tel calcul, il suffit de connaître l'opérateur de transmission ou l'opérateur impédance de surface entre les milieux élastique et fluide. Ceci évite l'identification des paramètres (densité et coefficients de Lamé) du milieu élastique. Nous définissons une classe d'opérateurs impédance approchés dans laquelle nous cherchons à identifier la meilleure approximation de l'impédance réelle du milieu élastique au sens où la pression correspondante est la plus proche de la pression exacte, au sens des moindres carrés dans une region d'observation. Nous traitons ici le problème direct associé à ce problème inverse. Nous étudions la stabilité du problème aux limites associé et en particulier sa résolution numérique. Deux schémas numériques sont décrits et un résultat de stabilité est démontré. Nous montrons quelques resultats de simulation. Pour le cas particulier d'une seule couche acoustique, l'impédance approchée correspond à une condition au bord absorbante. Notre travail s'applique donc à la discrétisation de conditions absorbantes d'ordre arbitraire.

Mots clés:

Opérateur impédance, interface acoustique-élastique, stabilité de schémas aux différences finies, identification, conditions aux limites, propagation d'ondes.

Key Words:

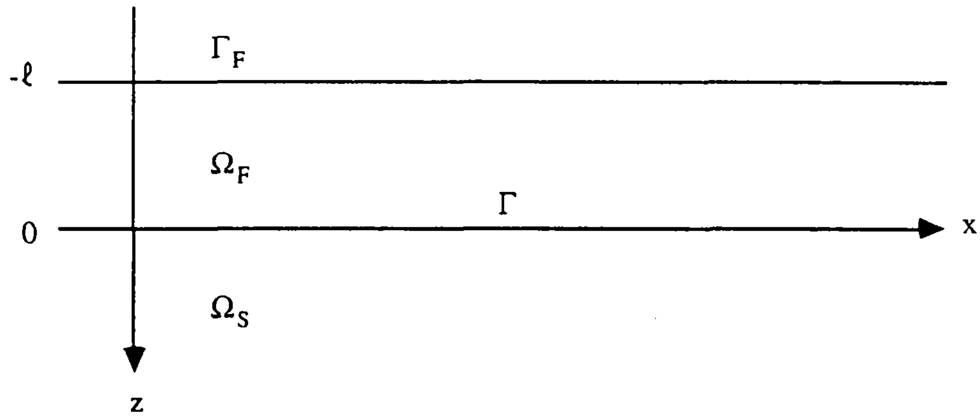
Impedance operator, acoustic-elastic interface, stability of finite difference schemes, identification, boundary conditions, wave propagation.

1 - Introduction

In certain physical problems one is interested in solving the wave equation in some part of the ocean with the objective of identifying a foreign body. However one must first identify the parameters characterizing the earth underneath the ocean floor, i.e. the density and the Lame coefficients. The idea that motivates this study is that it might be interesting to try to find instead of these usual parameters, the impedance of the ocean floor, that is the operator Z which associates to the time derivative of the fluid pressure at the ocean floor its outward normal derivative. However as one should suspect, Z is rarely if ever a differential operator. Thus we are led to define a class of approximate impedance operators which are differential operators and which lead to well posed problems when used to define an initial boundary value problem. It is in this class of operators that one would like to identify the example for which the solution to the associated initial boundary value problem best approximates an observed solution. Here we shall not treat in any detail this identification or inverse problem but shall be concerned with the direct problem. We define a class of approximate impedances and study the corrresponding initial boundary value problem, its well posedness and numerical resolution.

In the remainder of section 1 we give a mathematical description of the physical problem. In section 2 we define the impedance operator and in section 3 we give a class of approximate impedance operators. Section 4 is concerned with the stability of the continuous direct problem associated with an approximate impedance operator, and section 5 with its numerical approximation. Two numerical schemes are described and a numerical stability result is demonstrated. Numerical results are presented in section 6. A conclusion and a brief sketch of the inverse problem make up section 7.

Consider a fluid medium Ω_F overlying a stratified elastic medium Ω_S with interface Γ. Ω_S will be identified with the half plane $\{(x,z)\in \mathbb{R}^2 ; z > 0\}$ and Ω_F with the horizontal strip $\{(x,z)\in \mathbb{R}^2 ; -\ell < z < 0\}$. Γ is then the x-axis and the boundary $\Gamma_F = \{(x,z)\in \mathbb{R}^2 ; z = -\ell\}$ is a free boundary.



Behavior in Ω_F is governed by the acoustic wave equation :

(1.1) $$\rho_F \frac{\partial^2 p}{\partial t^2} - \lambda_F \Delta p = f, \qquad \text{in } \Omega_F$$

where p denotes the fluid pressure, ρ_F the density, λ_F the Lame parameter, and f a source term with compact support in Ω_F. The parameters ρ_F and λ_F are taken to be constant and the source term f depends on x, z, and t. In Ω_S the controling equation is the equation of linear elastodynamics :

(1.2) $$\rho \frac{\partial^2 u}{\partial t^2} - A(\lambda,\mu)u = 0, \qquad \text{in } \Omega_S.$$

The operator $A(\lambda,\mu)$ is just the divergence of the stress tensor $\sigma(u)$:

(1.3) $$A(\lambda,\mu) = \operatorname{div} \sigma(u)$$

where the stress tensor σ is given by :

(1.4) $$\sigma(u) = \begin{pmatrix} (\lambda+2\mu)\frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_z}{\partial z} & \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) & \lambda \frac{\partial u_x}{\partial x} + (\lambda+2\mu)\frac{\partial u_z}{\partial z} \end{pmatrix}$$

with u the displacement, ρ the density, and λ and μ the Lame coefficients. The parameters ρ,λ and μ are assumed to be functions of z alone ; Ω_S is stratified. The transmission conditions on Γ are the continuity of the normal component of displacement ; which leads to :

(1.5) $$-\frac{\partial p}{\partial n} = \rho \frac{\partial^2 u}{\partial t^2}.n \qquad \text{on } \Gamma$$

and the continuity of the stress times n :

(1.6) $$\sigma(u)n = -pn \qquad \text{on } \Gamma$$

where n is a unit vector normal to Γ.

The boundary condition on the free boundary Γ_F is :

(1.7) $$p = 0 \qquad \text{on } \Gamma_F.$$

Thus the transmission problem to be solved in $\Omega = \Omega_F \cup \Omega_S$ is :

(1.8) $$\begin{aligned} &\rho_F \frac{\partial^2 p}{\partial t^2} - \lambda_F \Delta p = f && \text{in } \Omega_F \\ &\rho \frac{\partial^2 u}{\partial t^2} - A(\lambda,\mu)u = 0 && \text{in } \Omega_S \\ &-\frac{\partial p}{\partial n} = \rho \frac{\partial^2 u}{\partial t^2}.n && \text{on } \Gamma \\ &-\sigma(u)n = pn && \text{on } \Gamma \\ &p = 0 && \text{on } \Gamma_F \\ &\text{initial conditions} = 0 && \text{on } \Omega_F \cup \Omega_S. \end{aligned}$$

2 - The impedance operator Z

The impedance operator Z associates to the time derivative of the fluid pressure on Γ, the negative of its outward normal derivative there, so that once Z is known, the pressure in Ω_F can be determined without further regard to the underlying elastic medium Ω_S. To define Z, let $\varphi(x,t)$ be a function defined on Γ and denote by $u^{\varphi}(x,z,t)$ the solution u of the following problem defined on Ω_S :

$$
(2.1)\qquad
\begin{aligned}
&\rho \frac{\partial^2 u}{\partial t^2} - \operatorname{div}(\sigma(u)) = 0 && \text{in } \Omega_S \\
&-\frac{\partial \sigma(u) n}{\partial t} = \varphi n && \text{on } \Gamma \\
&\text{zero initial conditions} && \text{on } \Omega_S.
\end{aligned}
$$

Then put :

$$Z(\varphi)\,(x,t) = \rho \frac{\partial^2 u^{\varphi}}{\partial t^2} .\, n\,(x,0,t),$$

so that Z is the composite operator :

$$
\begin{tikzcd}
\varphi \arrow[r] \arrow[rr, bend right, "Z"'] & u^{\varphi} \arrow[r] & -\rho \dfrac{\partial^2 u^{\varphi}}{\partial t^2} .n
\end{tikzcd}
$$

Now the solution p of (1.8) in Ω_F coincides with the solution of the following initial boundary value problem in Ω_F:

$$
(2.2)\qquad
\begin{aligned}
&\frac{\partial^2 p}{\partial t^2} - \Delta p = f && \text{in } \Omega_F \\
&\frac{\partial p}{\partial n} + Z\left(\frac{\partial p}{\partial t}\right) = 0 && \text{on } \Gamma \\
&p = 0 && \text{on } \Gamma_F \\
&\text{zero initial conditions} && \text{in } \Omega_F .
\end{aligned}
$$

Using the Fourier transform in the tangential variable x and the time variable t,

$$\varphi(x,t) \xrightarrow{\mathcal{F}} \hat{\varphi}(k,\omega),$$

k representing the wave number in the direction x and ω the pulsation, we define the complex impedance $\hat{Z}(k,\omega)$ of the stratified medium Ω_S so that :

$$\widehat{Z\varphi}(k,\omega) = \hat{Z}(k,\omega)\hat{\varphi}(k,\omega)$$

as follows :

Let $\hat{u}(k,z,\omega)$ be the solution $\hat{u}$ of the following system of ordinary differential equations :

$$- \rho\omega^2 \hat{u} - \widehat{\operatorname{div}\sigma(u)} = 0 \qquad \text{for } z > 0$$

(2.3)

$$- i\omega\widehat{\sigma(u)n} = n \qquad \text{at } z = 0,$$

or more explicitly

$$-\frac{d}{dz}\left(\mu(z)\frac{d\hat{u}_x}{dz}\right) - ik\left(\lambda\frac{d\hat{u}_z}{dz} + \frac{d}{dz}\,\mu\hat{u}_z)\right) + \left[(\lambda+2\mu)k^2 - \rho\omega^2\right]\hat{u}_x = 0 \qquad \text{for } z > 0$$

$$-\frac{d}{dz}\left((\lambda+2\mu)\frac{\partial\hat{u}_z}{\partial z}\right) - ik\left(\mu\frac{d\hat{u}_x}{dz} + \frac{d}{dz}\lambda\hat{u}_x)\right) + [\mu k^2 - \rho\omega^2]\hat{u}_z = 0 \qquad \text{for } z > 0$$

(2.4)

$$\mu\frac{d\hat{u}_x}{dz} + ik\hat{u}_z = 0 \qquad \text{at } z = 0$$

$$- i\omega\left((\lambda+2\mu)\frac{d\hat{u}_z}{dz} + ik\lambda\hat{u}_x\right) = 1 \qquad \text{at } z = 0$$

where u_x and u_z are the x and z components of u. Then the complex impedance $\hat{Z}(k,\omega)$ is given by

(2.5) $\qquad \hat{Z}(k,\omega) = - \rho\omega^2\hat{u}_z(k,0,\omega).$

If we introduce the function K(x,t) whose Fourier transform is $\hat{Z}(k,\omega)$,

$$K(x,t) \xrightarrow{\mathcal{F}} \hat{Z}(k,\omega),$$

the impedance operator is none other than the convolution operator

$$(Z\varphi)(x,t) = \int_0^{+\infty}\int_{-\infty}^{+\infty} K(x-\xi,t-\tau)\varphi(\xi,\tau)\, d\xi\, d\tau$$

with kernel K(x,t).

Examples

In the simple case for which the stratified medium Ω_S is in fact homogeneous we can calculate directly the impedance operator Z or more precisely the complex impedance $\hat{Z}(k,\omega)$ of Ω_S. Even though in the model in which we are interested Ω_S is an elastic medium we think it instructive to consider an example where Ω_S is another acoustic medium.

Ω_S a homogeneous acoustic medium

The parameters characterising Ω_F are ρ_F and λ_F and the pressure is denoted by p while the parameters characterizing Ω_S are ρ_S and λ_S and the pressure is p_s. We assume Ω_S and Ω_F are homogeneous, i.e. ρ_F,λ_F,ρ_S and λ_S are constant, and we denote the velocities in Ω_F and Ω_S by c_F and c_S respectively, $c^2{}_F = \lambda_F/\rho_F$ and $c^2{}_S = \lambda_S/\rho_S$.

Ω_F — pressure p, velocity c_F, Lamé coefficient λ_F

Γ ————————————→

Ω_S — pressure p_S, velocity c_S, Lamé coefficient λ_S

In this case the transmission conditions on Γ are

$$p = p_S \qquad \text{on } \Gamma \tag{2.6}$$

$$\lambda_F \frac{\partial p}{\partial z} = \lambda_S \frac{\partial p_S}{\partial z} \qquad \text{on } \Gamma, \tag{2.7}$$

and the impedance operator Z is given by

$$Z(\varphi)(x,t) = -\frac{\lambda_S}{\lambda_F} \frac{\partial p_S}{\partial z}(x,0,t), \tag{2.8}$$

where p_S is the solution in Ω_S of

$$\begin{cases} \dfrac{\partial^2 p_S}{\partial t^2} - c_S^2 \Delta p_S = 0 & \text{in } \Omega_S \\ \dfrac{\partial p_S}{\partial t} = \varphi & \text{on } \Gamma \\ \text{zero initial conditions} & \text{in } \Omega_S. \end{cases} \tag{2.9}$$

The complex impedance $\hat{Z}(k,\omega)$ is then given by

$$\hat{Z}(k,\omega) = -\frac{\lambda_S}{\lambda_F} \frac{\partial \hat{p}_S}{\partial z}(k,0,\omega)$$

with p_S the solution of the ordinary differential equation

$$\begin{cases} \dfrac{d^2 \hat{p}_S}{\partial z^2} + \dfrac{\omega^2}{c_S^2}\left(1 - \dfrac{c_S^2 k^2}{\omega^2}\right)\hat{p}_S = 0 & z > 0 \\ i\omega \hat{p}_S(k,0,\omega) = 1 & z = 0. \end{cases} \tag{2.10}$$

Solving (2.10) we obtain

$$\hat{p}_S(k,z,\omega) = \hat{p}_S(k,0,\omega)\, e^{-\frac{i\omega}{c_S}\sqrt{1-\left(\frac{c_S k}{\omega}\right)^2}\, z}$$

and

$$\hat{Z}(k,\omega) = \frac{1}{c_S} \frac{\lambda_S}{\lambda_F} \sqrt{1 - \left(\frac{c_S k}{\omega}\right)^2}. \tag{2.11}$$

Ω_S a homogeneous elastic medium

We retain the notation of section 1 and assume that $\rho_F, \lambda_F, \rho, \lambda$ and μ are constant. Let v_p and v_S denote the velocities of the p waves and s waves respectively in Ω_S,

$$v^2_p = (\lambda+2\mu)/\rho \qquad v^2_S = \mu/\rho.$$

Ω_F — pressure p_S; density ρ_F; velocity c; Lamé coefficient λ_F

$\Gamma \longrightarrow$

Ω_S — displacement u; density ρ; velocity of p waves v_p; velocity of s waves v_s; Lamé coefficients λ, μ

With Ω_S homogeneous, on dividing by ρ, the system of ordinary differential equations (2.4) becomes

$$v_s^2 \widehat{\frac{\partial^2 u_x}{\partial z^2}} + ik\,(v_p^2 - v_s^2)\,\widehat{\frac{\partial u_z}{\partial z}} - (k^2 v_p^2 - \omega^2)\,\hat{u}_x = 0 \qquad z > 0,$$

$$v_p^2 \widehat{\frac{\partial^2 u_z}{\partial z^2}} + ik\,(v_p^2 - v_s^2)\,\widehat{\frac{\partial u_x}{\partial z}} - (k^2 v_s^2 - \omega^2)\,\hat{u}_z = 0 \qquad z > 0,$$

$$v_s^2 \left(\widehat{\frac{\partial u_x}{\partial z}} + ik\hat{u}_z \right) = 0 \qquad z = 0,$$

$$-\rho i\omega \left(v_p^2 \widehat{\frac{\partial u_x}{\partial z}} + ik\,(v_p^2 - 2v_s^2)\,\hat{u}_z \right) = 1 \qquad z = 0.$$

The solution is thus of the form

$$\hat{u}_x = A(k,\omega)\,(-k)\,e^{-iy_p z} + B(k,\omega)\,y_s\,e^{-iy_s z}$$
$$\hat{u}_z = A(k,\omega)\,y_p\,e^{-iy_p z} + B(k,\omega)\,k\,e^{-iy_s z},$$

with :

(2.12) $$y_s(k,\omega) = \frac{\omega}{v_s}\sqrt{1-\left(\frac{kv_s}{\omega}\right)^2} \text{ and } y_p(k,\omega) = \frac{\omega}{v_p}\sqrt{1-\left(\frac{kv_p}{\omega}\right)^2},$$

and the constants $A(k,\omega)$ and $B(k,\omega)$ are calculated using the equations on z=0 :

$$A(k,\omega) = \frac{(\omega^2 - 2k^2 v_s^2)}{\Delta\omega\rho}, \quad B(k,\omega) = \frac{2kv_s^2 y_p}{\Delta\omega\rho}$$

with

(2.13) $$\Delta(k,\omega) = -\left[(\omega^2 - 2v_s^2 k^2)^2 + 4k^2 v_s^4 y_s y_p\right].$$

Hence we have

$$\widehat{u}_z(k,0,\omega) = \frac{\omega y_p(k,\omega)}{\rho\Delta(k,\omega)},$$

and (2.5) becomes

(2.14) $$\widehat{Z}(k,\omega) = \frac{\omega^3 y_p(k,\omega)}{\Delta(k,\omega)}.$$

Some properties of the operator Z

We state here without proof some of the properties of the operator Z which we shall try to preserve when we define a class of approximate impedance operators.

(1) Z commutes with the translation in space and translation in time operators :

$$Z(\varphi(x+L,t+\tau)) = (Z\varphi)(x+L,t+\tau).$$

In other words Z is a convolution operator.

(2) Z commutes with the symmetry operator in the variable x :

$$Z(\varphi(-x,t)) = (Z\varphi)(-x,t).$$

(3) Z is positive :

$$\int_0^T\int_{-\infty}^{\infty} (Z\varphi)(x,t)\varphi(x,t)\,dx\,dt \geq 0$$

for all $T > 0$. This condition assures us that the initial boundary value problem (2.2) is well posed. This property may be expressed in terms of the complex impedance $\hat{Z}$ as follows :

$$\mathrm{Re}(\hat{Z}(k,\omega)) \geq 0 \qquad \text{for all } (k,\omega) \in \mathbb{R}^2.$$

(4) Z is causal :

$$\varphi(x,t) = 0 \text{ for all } t \leq T \quad \Rightarrow \quad (Z\varphi)(x,T) = 0.$$

In terms of $\hat{Z}$ this property is that the function $\omega \to \hat{Z}(k,\omega)$ may be extended analytically to the complex half plane $\mathrm{Im}\omega > 0$.

(5) Z is real :

$$\varphi(x,t) \in \mathbb{R} \text{ for all } (x,t) \quad \Rightarrow \quad (Z\varphi)(x,t) \in \mathbb{R} \text{ for all } (x,t).$$

The translation of this property into terms of $\hat{Z}$ is :

$$\widehat{Z}(-k,\omega) = \widehat{Z}(k,-\omega) = \overline{\widehat{Z}(k,\omega)}.$$

We shall see in the next section that we can easily preserve all of the above properties for the class of approximate impedances that we shall define with the exception of (3), positivity. This property shall be replaced by the weaker property that the initial boundary value problem (2.2) corresponding to an approximate impedance operator be well posed.

3 - A class of approximate impedance operators

We denote by Σ the class of impedance operators Z corresponding to a stratified medium Ω_S and by $\hat{\Sigma}$ the class of the associated impedances $\hat{Z}$. Given that the ultimate objective is to solve an identification problem in a class Σ_a of approximate impedance operators, this class Σ_a should be constructed in such a way as to satisfy the following criteria :

1) Each element of Σ_a should be determined by a "small" number of parameters.
2) The class Σ_a should be large enough to "well approximate" all elements of Σ.
3) Each element of Σ_a should lead to a well posed initial boundary value problem.
4) The solution of the initial boundary value problem corresponding to an element of Σ_a should be "simple".

Evidently criteria 1) and 2) are at odds. From a practical point of view the reduction of the number of parameters is necessary in order to have a plausible numerical approach to the identification problem. Thus we have favored 1) over 2). Criterion 3), which obviously can not be violated, also goes against criterion 2) in that it restricts the set Σ_a. We shall see that it is indeed this criterion that limits us in our attempt to satisfy 2). Finally criterion 4) is equally necessary for the feasibility of our program in that the use of an optimization algorithm requires a succession of resolutions of the direct problem.

In fact it is criterion 4) that has guided us in our choice of Σ_a. In general the difficulty in working directly with an operator Z in Σ comes from the nonlocal nature in space and in time of these operators, which makes their numerical treatment prohibitif. However it is well known that certain convolution operators are easily calculated by solving a system of differential equations. Such is the case when the kernel of convolution is a sum of exponential functions or equivalently when its fourier transform, here the complex impedance $\hat{Z}(k,\omega)$, is a rational function of (k,ω), whence the idea to choose Σ_a as a subclass of the operators whose complex impedance is a rational function.

This idea has already been exploited by Engquist and Majda [2] and [3] for constructing absorbing boundary conditions. In fact, the problem of absorbing boundary conditions appears as a special case of the problem treated here in which the medium Ω_S is identical in nature to the medium Ω_F. The impedance complex given by (2.11) is :

$$\hat{Z}(k,\omega) = \frac{1}{c}\sqrt{1-\left(\frac{ck}{\omega}\right)^2},$$

$c = c_S = c_F$. In this case a method that has proven quite effective is to use the Pade approximations of the function $\sqrt{1-x^2}$ around $x = 0$. These approximations are given by

$$\sqrt{1-x^2} \simeq 1 - \sum_{p=1}^{N} \frac{a_p x^2}{\alpha_p x^2 + 1} \qquad (3.1)$$

where for the approximations of odd order, 2N+1, the coefficients are given by

$$a_p = \frac{2}{2N+1}\sin^2\frac{p\pi}{2N+1}$$

(3.2)

$$\alpha_p = -\cos^2\frac{p\pi}{2N+1},$$

and for the approximations of even order, 2N, the coefficients are given by

$$a_p = \frac{1}{N}\sin^2\frac{p\pi}{2N},\quad p = 1,\dots,N-1\ ;\ a_N = \frac{1}{2N}$$

(3.3)

$$\alpha_p = -\cos^2\frac{p\pi}{2N},\quad p = 1,\dots,N-1\ ;\ \alpha_N = 0.$$

It is natural to demand that the class Σ_a contain all operators Z whose corresponding impedance complex $\hat{Z}$ is of the form (3.1), and we have chosen to take Σ_a to be contained in the class of operators whose complex impedance is of the following form

$$\hat{Z}(k,\omega) = \frac{1}{c}\left(1 - \sum_{p=1}^{N}\frac{a_pk^2-b_p\omega^2+c_pi\omega+d_p}{\alpha_pk^2-\beta_p\omega^2+\gamma_pi\omega+\delta_p}\right), \tag{3.4}$$

with real coefficients $c,a_p,b_p,c_p,d_p,\alpha_p,\beta_p,\gamma_p,\delta_p$, $p=1,\dots,N$.

Several comments on the form (3.4) are in order. By demanding that the coefficients be real we respect the reality, property (5), of the true impedance. By construction we have respected property (1), invariance under translation, and property (4), causality. The spatial symmetry, property (2), is guaranteed by the fact that the rational function $\hat{Z}(k,\omega)$ is an even function of k. We have required that the "integer part" of the rational function be constant. This is practically required in order to obtain a well posed problem. We shall see in the next section what other requirements must be imposed on the coefficients to obtain a well posed problem. Finally the choice of the form (3.4), with $\hat{Z}(k,\omega)$ given as a sum of simple elements, is justified by the expression of $Z(\varphi)$ in the physical variables (x,t) which makes the numerical resolution of the problem quite simple as we shall see in section 5.

We shall define Σ_a to be the class of operators Z such that :

(3.5) $\hat{Z}(k,\omega)$ is of the form (3.4)

(3.6) the associated inital boundary value problem (2.2) is well posed.

We denote by $\hat{\Sigma}_a$ the class of corresponding complex impedances.

4 - The direct problem

Given an 8N+1 tuple $m \in \mathbb{R}^{8N+1}$,

$$m = (c,(a_p,b_p,c_p,d_p,\alpha_p,\beta_p,\gamma_p,\delta_p),\ p=1,\dots,N),$$

we denote by $\hat{Z}_m$ the complex impedance of the form (3.4) corresponding to m and by Σ_m the impedance operator. The direct problem associated to m is the problem (2.2) for $Z=Z_m$:

$$(4.1)\qquad \begin{cases} \dfrac{\partial^2 p}{\partial t^2} - \Delta p = f & \text{in } \Omega_F \\ \dfrac{\partial p}{\partial n} + Z_m\left(\dfrac{\partial p}{\partial t}\right) = 0 & \text{on } \Gamma \\ p = 0 & \text{on } \Gamma_F \\ p = \dfrac{\partial p}{\partial t} = 0 & \text{in } \Omega_F \text{ at } t = 0. \end{cases}$$

For simplicity we have assumed $c_F = \lambda_F = \rho_F = 1$. The first question that arises in the study of such a problem is whether the problem is well posed. Does there exist a unique solution depending continuously on f. The answer of course depends on m and will give us an explicit criterion for determining which parameters $m \in \mathbb{R}^{8N+1}$ correspond to an element of Σ_a.

Given an $m \in \mathbb{R}^{8N+1}$, $\hat{Z}_m(k,\omega)$ is a rational function of (k,ω) and may be written in the form

$$(4.2)\qquad \hat{Z}_m(k,\omega) = \frac{Q(ik,i\omega)}{R(ik,i\omega)},$$

with
$$Q(ik,i\omega) = \prod_{p=1}^{N} (\alpha_p k^2 - \beta_p \omega^2 + \gamma_p i\omega + \delta_p)$$
$$+ \sum_{p=1}^{N} \left((a_p k^2 - b_p \omega^2 + c_p i\omega + d_p) \prod_{\substack{q=1 \\ q\neq p}}^{N} (\alpha_q k^2 - \beta_q \omega^2 + \gamma_q i\omega + \delta_q) \right)$$
$$R(ik,i\omega) = c \prod_{p=1}^{N} (\alpha_p k^2 - \beta_p \omega^2 + \gamma_p i\omega + \delta_p).$$

The polynomials Q and R have real coefficients and are of degree less than or equal to 2N, and we may write

$$R(ik,i\omega)\,\hat{Z}_m(k,\omega)\,\hat{\varphi} = Q(ik,i\omega)\hat{\varphi}.$$

Applying the inverse Fourier transform we obtain

$$R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) Z_m \varphi = Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\varphi$$

which leads to the following boundary condition

$$R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial n} + Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial t} = 0 \quad \text{on } \Gamma.$$

Indeed we impose that R and Q are of order 2N by assuming $\beta_p \neq 0$ and $a_p \neq 0$, $p=1,\dots,N$. We also assume that

$$\frac{\alpha_p}{\beta_p} \neq \frac{\alpha_q}{\beta_q} \text{ if } p\neq q.$$

Since the stability of the problem depends only on the terms of highest degree we introduce the homogeneous polynomials $\tilde{R}(ik,i\omega)$ and $\tilde{Q}(ik,i\omega)$ consisting of the terms of $R(ik,i\omega)$ and $Q(ik,i\omega)$ respectively of maximal degree :

$$\tilde{Q}(ik,i\omega) = \prod_{p=1}^{N} (\alpha_p k^2 - \beta_p \omega^2) + \sum_{p=1}^{N} \left((a_p k^2 - b_p \omega^2) \prod_{\substack{q=1 \\ q \neq p}}^{N} (\alpha_q k^2 - \beta_q \omega^2) \right)$$

$$\tilde{R}(ik,i\omega) = c \prod_{p=1}^{N} (\alpha_p k^2 - \beta_p \omega^2),$$

and consider the condition

$$\tilde{R}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial n} + \tilde{Q}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial t} = 0 \quad \text{on } \Gamma$$

and the problem

$$\begin{aligned} &\frac{\partial^2 p}{\partial t^2} - \Delta p = f && \text{in } \Omega = \{(x,z) : z < 0\} \\ &\tilde{R}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial n} + \tilde{Q}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial t} = 0 && \text{on } \Gamma \\ &p = \frac{\partial p}{\partial t} = 0 && \text{in } \Omega \text{ at } t = 0, \end{aligned} \tag{4.3}$$

where for simplicity we have replaced the domain Ω_F by the half space Ω.

The stability of the system (4.3) may be studied by the method of normal modes ; see [4], [5], and [6] for a detailed development of this theory. A normal mode is a solution of the equations

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = 0 \quad \text{in } \Omega \tag{4.4}$$

$$\tilde{R}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial n} + \tilde{Q}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial t} = 0 \quad \text{on } \Gamma \tag{4.5}$$

of the form

$$p = e^{(h+ik)x + (\ell+i\lambda)z + (\eta+i\omega)t} = e^{hx+\ell y+\eta t} e^{i(kx+\lambda z+\omega t)}$$

with amplitude $e^{hx+\ell y+\eta t}$, frequency ω, and velocity vector $v = (v_x,v_z) = (-k/\omega,-\lambda/\omega)$. Since p satisfies (4.4) we have the dispersion relation

$$(\eta+i\omega)^2 - (h+ik)^2 - (\ell+i\lambda)^2 = 0,$$

and since for t fixed, p should remain bounded in Ω we impose h=0 and $\ell \geq 0$. Thus we have

$$\ell+i\lambda = \sqrt{(\eta+i\omega)^2+k^2} . \tag{4.6}$$

As p is also a solution of (4.5) we also have

$$\tilde{R}(ik,\eta+i\omega)(\ell+i\lambda) + \tilde{Q}(ik,\eta+i\omega)(\eta+i\omega) = 0. \tag{4.7}$$

Combining (4.6) with (4.7) we obtain the so called *characteristic equation* of problem (4.3) :

$$F(k,\eta,\omega) = \sqrt{k^2+(\eta+i\omega)^2}\, \tilde{R}(ik,\eta+i\omega)+(\eta+i\omega)\, \tilde{Q}(ik,\eta+i\omega) = 0. \tag{4.8}$$

It is in terms of the zeros of this equation that we can describe the well posedness of problem (4.3).

Problem (4.3) and hence (4.1) is *strongly well posed* if there is no solution of (4.8) with $\eta \geq 0$. It is *weakly well posed* if there is no solution with $\eta > 0$. It is *strongly ill posed* if there exists a solution with $\eta > 0$. For a strongly well posed problem the energy in Ω at any time $T > 0$ can be bounded in the L_2-norm of the source term f. For a weakly well posed problem the energy can be bounded in terms of a higher order Sobolev norm of f, the order needed depending on the multiplicity of the root with $\eta = 0$. One says in this case that derivatives are lost. Finally, for an ill posed problem the energy in Ω grows in time at a completely uncontroled rate. For precise statements of these estimates we refer the reader to [5] or [6].

Before translating these conditions into terms of the parameters m we make a heuristic comment on the numerical approximation of problem (4.1), or (4.3). Denote by $\Sigma^S{}_a$ the set of operators Z for which $\hat{Z}$ is of the form (3.4) and for which the problem (4.3) is strongly well posed and by $\Sigma^W{}_a$ the set of operators Z for which $\hat{Z}$ is of the form (3.4) and for which the problem (4.3) is weakly well posed. The set $\Sigma^S{}_a$ of operators having the zeros of its characteristic equation (4.8) in the open set $\eta < 0$ is open. $\Sigma^W{}_a$ is the boundary between $\Sigma^S{}_a$ and the set of operators giving rise to an ill posed problem. This indicates that the numerical approximation of a weakly well posed problem is more delicate than that of a strongly well posed problem. A "good" approximation of a strongly well posed problem should be stable, where as a weakly well posed problem may be approached by a numerical problem that is stable or by a numerical problem that is unstable.

To describe the subset of $\mathbb{R}^{8N+1}$ of parameters which correspond to an operator giving rise to a strongly or weakly well posed problem we use a characterization of well posedness developed in [8]. Since $\tilde{R}$ and $\tilde{Q}$ are homogeneous of the same degree, (4.7) implies that

$$\ell+i\lambda = -\frac{\tilde{Q}\left(\frac{ik}{\eta+i\omega},1\right)}{\tilde{R}\left(\frac{ik}{\eta+i\omega},1\right)}(\eta+i\omega).$$

Defining

$$r(\theta) = \frac{\tilde{Q}(\theta,1)}{\tilde{R}(\theta,1)}, \quad \theta \in \mathbb{C}, \tag{4.9}$$

we may write

$$\ell+i\lambda = -\frac{r(\theta)}{\theta}ik, \text{ with } \theta = \frac{ik}{\eta+i\omega}.$$

We may now characterize the strong or weak well posedness of (4.1) and (4.3) in terms of the poles and zeros of $r(\theta)/\theta$.

Problem (4.3) and hence (4.1) is strongly well posed if the zeros and poles of $r(\theta)/\theta$ are real, simple and alternate along the real axis and $r(\theta) > 0$, for $\theta \in [-1,1]$. It is weakly well posed if the zeros and poles of $r(\theta)/\theta$ are real, simple and alternate along the real axis and $r(0) > 0$. (cf. [8], theorems 1 and 2).

Decomposing $r(\theta)/\theta$ into partial fractions we obtain

$$\frac{r(\theta)}{\theta} = \psi_*\theta + \psi_0\frac{1}{\theta} + \sum_{\{p;\alpha_p\neq 0\}} \psi_p\left(\frac{1}{\theta-\theta_p} + \frac{1}{\theta+\theta_p}\right)$$

with

$$\psi_* = \frac{1}{c} \sum_{\{p;\alpha_p=0\}} \frac{a_p}{\beta_p}$$

$$\psi_0 = \frac{1}{c}\left\{1 - \sum_{p=1}^{N} \frac{b_p}{\beta_p}\right\}$$

$$\psi_p = -\frac{1}{2c}\left(\frac{a_p}{\alpha_p} - \frac{b_p}{\beta_p}\right) \quad \text{for all p with } \alpha_p \neq 0$$

$$\theta_p = \sqrt{\frac{\beta_p}{\alpha_p}} \qquad \text{for all p with } \alpha_p \neq 0.$$

In order that the poles of $r(\theta)/\theta$ be real and simple it is necessary and sufficient that β_p and α_p have the same sign. There will be exactly one zero between a pair of consecutive poles if and only if the coefficients ψ_p, p=0,...,N are of the same sign and ψ_* is of the opposite sign. Since the coefficient of $1/\theta$ determines the sign of r(0), the conditions for having a weakly well posed problem are

$$\text{(4.10)} \quad \begin{aligned} &\beta_p/\alpha_p > 0 \quad \text{for all p with } \alpha_p \neq 0 \quad p=1,\dots,N \\ &\psi_p > 0 \quad \text{for all p with } \alpha_p \neq 0 \quad p=1,\dots,N \\ &\psi_0 > 0 \\ &\psi_* \leq 0. \end{aligned}$$

To have further that $r(\theta) > 0$ for $-1 \leq \theta \leq 1$ we need in addition to the conditions (4.10) that no pole of $r(\theta)$ lies in $[-1,1]$ and that $r(1) > 0$:

(4.11) $\quad \beta_p/\alpha_p > 1 \qquad$ for all p with $\alpha_p \neq 0$; p=1,...,N

$$\text{(4.12)} \quad r(1) = \frac{1}{c}\left(1 - \sum_{p=1}^{N} \frac{a_p - b_p}{\alpha_p - \beta_p}\right) > 0.$$

Remark 4.1

Suppose that m is such that (4.10) and (4.11) hold but instead of (4.12) we have

$$\text{(4.13)} \quad r(1) = \frac{1}{c}\left(1 - \sum_{p=1}^{N} \frac{a_p - b_p}{\alpha_p - \beta_p}\right) = 0.$$

Then problem (4.1), (4.3) is only weakly well posed. However, the solution (k,η,ω) of the characteristic equation (4.8) with $\eta=0$ also has k=0. It is said to be a "generalized eigenvalue of the first type". In certain cases it has been shown that one can obtain interior energy estimates for such problems resembling those obtained for strongly well posed problems, cf. [7]. We shall see in our numerical examples of paragraph 6 that such a problem exhibits the behavior of a strongly well posed problem.

We consider the approximation of the true impedance operators Z calculated in paragraph 2, (2.11) and (2.14), by elements Z_m in $\Sigma^S{}_a$ and $\Sigma^W{}_a$.

The Pade approximations of (2.11) are given by (3.1) with (3.2) and (3.3) :

$$\widehat{Z}_{2N+1}(k,\omega) = \frac{1}{c}\left\{ 1 - \sum_{p=1}^{N} \frac{a_p k^2}{\alpha_p k^2 - \beta_p \omega^2} \right\}$$

$$c = c_S \frac{\lambda_F}{\lambda_1}$$

(4.14)

$$a_p = \frac{2}{2N+1}\sin^2\frac{p\pi}{2N+1}c_S^2 \qquad p=1,\dots,N$$

$$\alpha_p = -\cos^2\frac{p\pi}{2N+1}c_S^2 \qquad p=1,\dots,N$$

$$\beta_p = -1 \qquad p=1,\dots,N,$$

and :

$$\widehat{Z}_{2N}(k,\omega) = \frac{1}{c}\left\{ 1 - \sum_{p=1}^{N} \frac{a_p k^2}{\alpha_p k^2 - \beta_p \omega^2} \right\}$$

$$c = c_S \frac{\lambda_F}{\lambda_1}$$

(4.15)

$$a_p = \frac{1}{N}\sin^2\left(\frac{p\pi}{2N}\right)c_S^2 \qquad p=1,\dots,N-1, \quad a_N = \frac{1}{2N}c_1^2$$

$$\alpha_p = -\cos^2\left(\frac{p\pi}{2N}\right)c_S^2 \qquad p=1,\dots,N-1, \quad \alpha_N = 0$$

$$\beta_p = -1 \qquad p=1,\dots,N.$$

We can verify immediately that Z_{2N} and Z_{2N+1} belong to $\Sigma^W{}_a$, but it is also obvious that condition (4.11) can not be satisfied for large N if $c_S > c$, and Z_{2N} and Z_{2N+1} will not be in $\Sigma^S{}_a$.

We consider the approximations of lowest order

(4.16) $$\widehat{Z}_1(k,\omega) = \frac{\lambda_S}{\lambda_F c_S}$$

(4.17) $$\widehat{Z}_2(k,\omega) = \frac{\lambda_S}{\lambda_F c_S}\left(1 - \frac{\frac{1}{2}c_S^2 k^2}{\omega^2}\right)$$

(4.18) $$\widehat{Z}_3(k,\omega) = \frac{\lambda_S}{\lambda_F c_S}\left(1 - \frac{\frac{1}{2}c_S^2 k^2}{\omega^2 - \frac{1}{4}c_S^2 k^2}\right),$$

and the corresponding functions $r_N(\theta)$

$$r_1(\theta) = [\lambda_S/(\lambda_F c_S)]$$

(4.19) $$r_2(\theta) = [\lambda_S/(\lambda_F c_S)](1-1/2\, c^2{}_S\theta^2)$$

$$r_3(\theta) = [\lambda_S/(\lambda_F c_S)][1-(1/2\, c^2{}_S\theta^2)/(1-1/4\, c^2{}_S\theta^2)].$$

Since the constants $\lambda_S\,\lambda_F$ and c_S are always positive $r_1(\theta)$ is always positive and $Z_1 \in \Sigma^S{}_a$ but $r_2(1)$ is negative once $c_S > \sqrt{2}$ and $r_3(1)$ is negative once $c_S > (2\sqrt{2})/3$.

The function $r(\theta)$ corresponding to the true impedance $\hat{Z}(k,\omega)$ given by (2.11) is :

(4.20) $$r(\theta) = \frac{\lambda_S}{c_S\lambda_F}\sqrt{1-c_S^2\theta^2}\ ,$$

whose graph is a semi-ellipse with axes of lengths $\lambda_S/(\lambda_F c_S)$ and $1/c_S$. It is clear that if $c_S >> 1$ we can not approximate $r(\theta)$ well for large values of θ by functions $r_a(\theta)$ which remain positive in the interval [-1,1]. We can find approximations $r_a(\theta)$ better that $r_1(\theta)$ which are positive in [-1,1], for example :

(4.21) $$r_a(\theta) = \frac{\lambda_S}{c_S\lambda_F}\,(1-(1-\varepsilon)\theta),$$

but they will not be more than first order at $\theta=0$ and will not be good approximations for $\theta=1$ or -1, cf.Figure 1. This is why we are interested in the class $\Sigma^W{}_a$ as well as the class $\Sigma^S{}_a$.

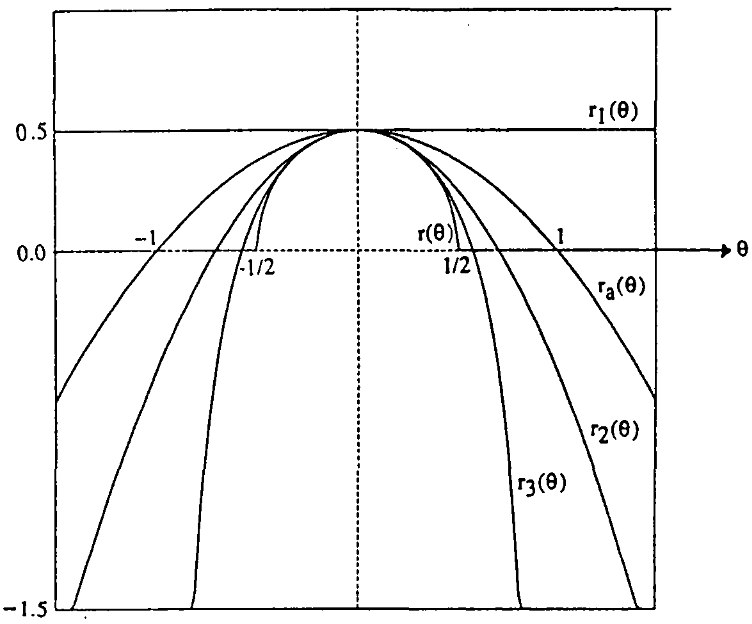


Figure 1. The functions $r(\theta)$, $r_1(\theta)$, $r_2(\theta)$, $r_3(\theta)$, and $r_a(\theta)$.

A general expression for the Pade approximations of (2.14) is not so readily obtained. We give the expressions for the approximations of lowest order

$$\hat{Z}_1(k,\omega) = \frac{1}{v_p}$$

(4.22) $$\hat{Z}_2(k,\omega) = \frac{1}{v_p}\left(1 - \frac{ak^2}{\omega^2}\right)$$

$$\hat{Z}_3(k,\omega) = \frac{1}{v_p}\left(1 - \frac{ak^2}{\omega^2 - \alpha k^2}\right)$$

and the corresponding functions $r_N(\theta)$

$$r_1(\theta) = \frac{1}{v_p}$$

$$(4.23) \qquad r_2(\theta) = \frac{1}{v_p}(1 - a\theta^2)$$

$$r_3(\theta) = \frac{1}{v_p}\left(1 - \frac{a\theta^2}{1-\alpha\theta^2}\right),$$

where

$$a = \left(\frac{v_p^2}{2} + 4\frac{v_s^3}{v_p} - 4v_s^2\right)$$

$$\alpha = \frac{1}{a}\left(\frac{v_p^4}{8} + 4\,v_s^4 - 2\frac{v_s^3}{v_p}\left(v_p^2+v_s^2\right)\right) - 4\,v_s^4\left(\frac{v_s}{v_p} - 1\right).$$

We observe that $Z_1 \in \Sigma^S{}_a$ for all ρ,λ,μ. Z_2 is in $\Sigma^W{}_a$ if $a \geq 0$ and $Z_2 \in \Sigma^S{}_a$ if $0 < a < 1$. However Z_2 gives a strongly ill posed problem if $a < 0$. We may write

$$(4.24) \qquad \frac{a}{v_p^2} = f(v) = \frac{1}{2} - 4v^2(1-v),$$

with $v = v_s/v_p = \sqrt{\mu/\lambda+2\mu}$. For all λ and μ, $0 \leq v \leq \sqrt{2/2}$.

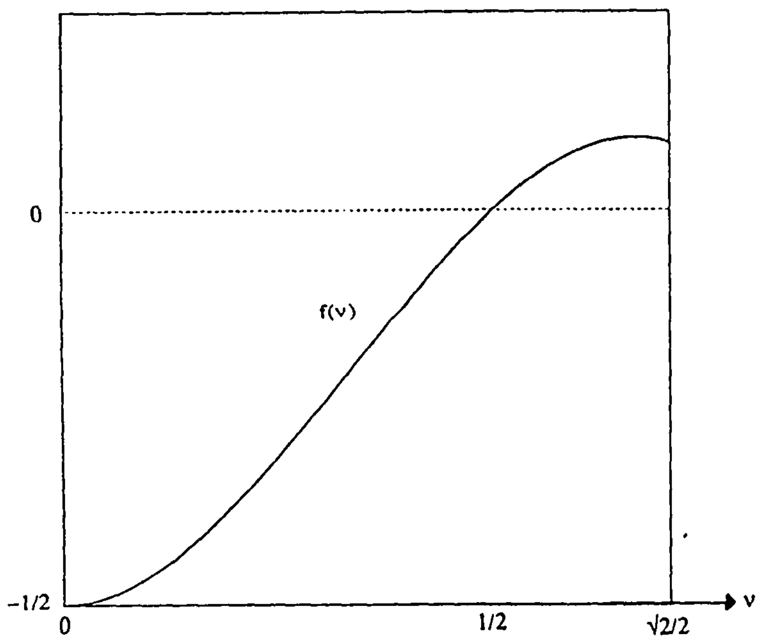


Figure 2. The function f(v).

We see that we have a strongly well posed problem if $0 < v < 1/2$ and $1/2 - 1/v^2_p < 4v^2(1-v)$ which is the case for instance if $0 < v < 1/2$ and $v_p < \sqrt{2}$. The problem is weakly well posed if $0 < v < 1/2$, but if $1/2 < v < \sqrt{2}/2$ we have an ill posed problem. For v in $[1/2, \sqrt{2}/2\,]$ the function $r(\theta)$ corresponding to (2.14) is concave upward at $\theta=0$ and thus can not be approximated to order greater than 1 at $\theta = 0$ even by a function $r(\theta)$ corresponding to a complex impedance $\hat{Z}(k,\omega) \in \hat{\Sigma}^W{}_a$. This is an example of where condition 3) of paragraph 3 restricts us in our attempt to satisfy condition 2), and this represents a certain limitation to the proposed procedure.

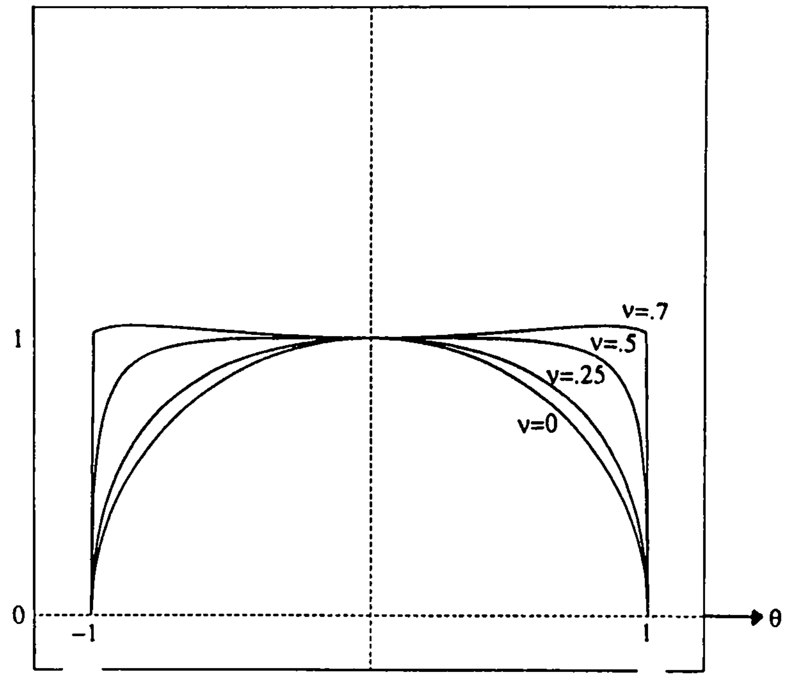


Figure 3. The functions $r(\theta)$ for $v_p=1$ and $v=0$, $1/4$, $1/2$, and $\sqrt{2}/2$.

5 - Numerical resolution of the direct problem

The form (3.4) for the complex impedance $\hat{Z}$ was chosen to facilitate the resolution of the direct problem (4.1). Indeed

$$(5.1)\qquad \left(\widehat{Z\frac{\partial p}{\partial t}}\right)(k,\omega) = \frac{i\omega}{c}\left\{\hat{p}(k,\omega) - \sum_{p=1}^{N}\left[\frac{a_pk^2-b_p\omega^2+c_pi\omega+d_p}{\alpha_pk^2-\beta_p\omega^2+\gamma_pi\omega+\delta_p}\right]\hat{p}(k,\omega)\right\}$$

is equivalent to the system

$$(5.2)\qquad \begin{cases} \left(\widehat{Z\frac{\partial p}{\partial t}}\right)(k,\omega) = \frac{i\omega}{c}\left\{\hat{p}(k,\omega) - \sum\limits_{p=1}^{N}\hat{\varphi}_p(k,\omega)\right\} \\ \left[\alpha_pk^2-\beta_p\omega^2+\gamma_pi\omega+\delta_p\right]\hat{\varphi}_p(k,\omega) = \left[a_pk^2-b_p\omega^2+c_pi\omega+d_p\right]\hat{p}(k,\omega), \end{cases}$$

where we have introduced the auxillary variables $\varphi_p(x,t)$ having Fourier transform $\hat{\varphi}_p(k,\omega)$. Applying the inverse Fourier transform to (5.2) we obtain

$$
(5.3)\qquad
\begin{aligned}
&Z\frac{\partial p}{\partial t}=\frac{1}{c}\left\{\frac{\partial p}{\partial t}-\sum_{p=1}^{N}\frac{\partial \varphi_p}{\partial t}\right\}\\
&\beta_p\frac{\partial^2\varphi_p}{\partial t^2}-\alpha_p\frac{\partial^2\varphi_p}{\partial x^2}+\gamma_p\frac{\partial \varphi_p}{\partial t}+\delta_p\varphi_p=b_p\frac{\partial^2 p}{\partial t^2}-a_p\frac{\partial^2 p}{\partial x^2}+c_p\frac{\partial p}{\partial t}+d_p p.
\end{aligned}
$$

Thus problem (4.1) is equivalent to

$$
(5.4)\qquad
\begin{aligned}
&\frac{\partial^2 p}{\partial t^2}-\Delta p=f && \text{in } \Omega_F\\
&p=0 && \text{on } \Gamma_F\\
&p=\frac{\partial p}{\partial t}=0 && \text{in } \Omega_F \text{ at } t=0\\
&\frac{\partial p}{\partial n}+\frac{1}{c}\left(\frac{\partial p}{\partial t}-\sum_{p=1}^{N}\frac{\partial \varphi_p}{\partial t}\right)=0 \quad \text{on } \Gamma\\
&\beta_p\frac{\partial^2\varphi_p}{\partial t^2}-\alpha_p\frac{\partial^2\varphi_p}{\partial x^2}+\gamma_p\frac{\partial \varphi_p}{\partial t}+\delta_p\varphi_p=b_p\frac{\partial^2 p}{\partial t^2}-a_p\frac{\partial^2 p}{\partial x^2}+c_p\frac{\partial p}{\partial t}+d_p p \quad \text{on } \Gamma\\
&\varphi_p=\frac{\partial \varphi_p}{\partial t}=0,\quad p=1,\dots,N, \qquad \text{on } \Gamma \text{ at } t=0,
\end{aligned}
$$

and it is for this problem that we shall construct a numerical scheme.

The scheme that we have used is a finite difference scheme on a regular grid of squares of side length h. The unknowns are the values p^n_{ij} of p at the points (ih,jh) in $\Omega_F \cup \Gamma$ at instant $n\Delta t$ and the values φ^n_{pi} of φ_p at (ih,Jh) on the boundary Γ, z=Jh, at instant $n\Delta t$, p=1,...,N.

The scheme for the equation in Ω_F is the standard five point scheme for the Laplacian and the explicit centered difference scheme for the second time derivative :

$$
(5.5)\qquad \frac{p_{ij}^{n+1}-2p_{ij}^{n}+p_{ij}^{n-1}}{\Delta t^2}-\frac{p_{ij+1}^{n}+p_{ij-1}^{n}+p_{i+1j}^{n}+p_{i-1j}^{n}-4p_{ij}^{n}}{h^2}=f_{ij}^{n}.
$$

For the equation on Γ we first considered a centered explicit scheme. The first and second time derivatives are approximated by standard centered differences and the second derivative in x is approximated by the standard centered difference :

$$
(5.6)\qquad \frac{\partial u}{\partial t}(ih,Jh,n\Delta t)\sim\frac{1}{2\Delta t}\left(u_{iJ}^{n+1}-u_{iJ}^{n-1}\right)
$$

$$
(5.7)\qquad \frac{\partial^2 u}{\partial t^2}(ih,Jh,n\Delta t)\sim\frac{1}{\Delta t^2}\left(u_{iJ}^{n+1}-2u_{iJ}^{n}+u_{iJ}^{n-1}\right)
$$

$$
(5.8)\qquad \frac{\partial^2 u}{\partial x^2}(ih,Jh,n\Delta t)\sim\frac{1}{h^2}\left(u_{i+1J}^{n}-2u_{iJ}^{n}+u_{i-1J}^{n}\right)
$$

where u is either p or φ_p. The normal derivative at Γ is approximated with a centered difference introducing fictitions values p^n_{iJ+1}

$$
(5.9)\qquad \frac{\partial p}{\partial z}(ih,Jh,n\Delta t)\sim\frac{1}{2h}\left(p_{i,J+1}^{n}-p_{i,J-1}^{n}\right),
$$

and these fictitions values are then eliminated by supposing that (5.5) holds on Γ, for $j=J$.

Thus we obtain N+1 equations with which we can calculate explicitly p^{n+1}_{iJ} and φ^{n+1}_{pi}, $p=1,\dots,N$, knowing all of the values, $p^{n}_{i'j}$ and $\varphi^{n}_{pi'}$, of p and φ_p on Γ and of p in Ω_F at the preceeding time $n\Delta t$.

We note that this scheme is identical to the one we would obtain using P_1 finite elements with mass lumping.

With this scheme we have obtained good numerical results for the case $Z_m \in \Sigma^S{}_a$, i.e. when the problem is strongly well posed ; see section 6. However in most of the cases for which the problem is only weakly well posed the scheme became unstable, the exception being those corresponding to the case of a generalized eigenvalue of the first kind, of Remark 4.1.

In order to calculate with impedance operators Z_m in $\Sigma^W{}_a$ we have devised a second numerical scheme. We keep the scheme (3.5) to calculate the interior values p^{n+1}_{ij}, $j < J$. In order to conserve the second order precision of the scheme we have kept a centered scheme, but this time it is centered at (ih,(J-1)h). For the first equation on Γ in (5.4), with only first order time derivatives, the approximation is centered in time at instant n+1/2. Thus the time derivative is approximated by the average between (ih,Jh) and (ih,(J-1)h) of the centered difference at instant n+1/2 :

$$(5.10) \qquad \frac{\partial u}{\partial t}(ih,(J-1/2)h,(n+1/2)\Delta t) \sim \frac{1}{\Delta t}\left(u_i^{n+1} - u_i^{n}\right),$$

where u^n_i is either $(p^n_{iJ}+p^n_{iJ-1})/2$ or φ^n_{pi}, and the normal derivative is approximated by the average between instants n+1 and n of the centered difference at (ih,(J-1/2)h) :

$$(5.11) \qquad \frac{\partial p}{\partial n}(ih,(J-1/2)h,(n+1/2)\Delta t) \sim \frac{1}{2}\left(\frac{1}{h}\left(p_{iJ}^{n+1} - p_{i,J-1}^{n+1}\right) + \frac{1}{h}\left(p_{iJ}^{n} - p_{i,J-1}^{n}\right)\right).$$

For the second equation on Γ in (5.4) with second order time derivatives, the approximation is centered in time at instant n. Thus the first and second time derivatives are approximated as in the previous scheme (5.6)-(5.8) except here we take the average of the centered difference at (ih,Jh) and the centered difference at (ih,(J-1)h) :

$$(5.12) \qquad \frac{\partial u}{\partial t}(ih,(J-1/2)h,n\Delta t) \sim \left(\frac{1}{2\Delta t}\left(u_i^{n+1} - u_i^{n-1}\right)\right)$$

$$(5.13) \qquad \frac{\partial^2 u}{\partial t^2}(ih,(J-1/2)h,n\Delta t) \sim \frac{1}{\Delta t^2}\left(u_i^{n+1} - 2u_i^{n} + u_i^{n-1}\right),$$

where u^n_i is either $(p^n_{iJ}+p^n_{iJ-1})/2$ or φ^n_{pi}, and for the second derivative in x we take the average of the centered difference at (ih,Jh) and at (ih,(J-1)h) and then we average this average calculated at instant n+1 and that calculated at instant n-1 :

$$(5.14) \qquad \frac{\partial^2 u}{\partial x^2}(ih,(J-1/2)h,n\Delta t) \sim \frac{1}{2}\left(\frac{1}{h^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) + \frac{1}{h^2}\left(u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}\right)\right).$$

The use of instant n+1 in (5.11) and (5.14) makes this scheme implicit on Γ. However, in the calculations we have done, this scheme has remained stable even for weakly well posed problems corresponding to an impedance operator in $\Sigma^W{}_a$, cf. Section 6.

In fact we have the following theorem which we shall prove for the simplest case in which we can encounter a weakly well posed problem ; i.e. when $\hat{Z}$ is of the following form :

$$\hat{Z}(k,\omega) = \frac{1}{c}\left(1 - \frac{ak^2}{\omega^2}\right),$$

and the boundary condition from (5.4)) on Γ is of the form

$$\frac{\partial p}{\partial z} = -\frac{1}{c}\left(\frac{\partial p}{\partial t} - \frac{\partial \varphi}{\partial t}\right)$$

$$(5.15)\qquad \frac{\partial^2 \varphi}{\partial t^2} = a\frac{\partial^2 p}{\partial x^2}$$

or equivalently

$$(5.16)\qquad \frac{\partial^2 p}{\partial t^2} + c\frac{\partial^2 p}{\partial t\partial z} - a\frac{\partial^2 p}{\partial x^2} = 0.$$

(We have weak stability if c and a are positive, strong stability if in addition $a < 1$).

Theorem 5.1.

Suppose we have the following problem :

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = f \qquad \text{for } z < 0$$

$$(5.17)\qquad \frac{\partial^2 p}{\partial t^2} + c\frac{\partial^2 p}{\partial t\partial z} - a\frac{\partial^2 p}{\partial x^2} = 0 \qquad \text{for } z = 0$$

$$p = \frac{\partial p}{\partial t} = 0 \qquad \text{for } z < 0 \text{ and } t = 0$$

with a and c positive real constants. Then for $1/4 < \theta \le 1/2$, the numerical scheme

$$(5.18)\qquad \Box_h p^n_{ij} = \frac{1}{\Delta t^2}\left(p^{n+1}_{ij} - 2p^n_{ij} + p^{n-1}_{ij}\right) - \frac{1}{h^2}\left(p^n_{ij+1} + p^n_{i,j-1} + p^n_{i+1,j} + p^n_{i-1,j} - 4p^n_{ij}\right) = f^n_{ij} \quad j = -\infty,\ldots,-1$$

$$(5.19)\qquad \frac{1}{\Delta t^2}\left(p^{n+1}_{i,-1/2} - 2p^n_{i,-1/2} + p^{n-1}_{i,-1/2}\right) + \frac{c}{2\Delta t}\left(\frac{p^{n+1}_{i0} - p^{n+1}_{i,-1}}{h} - \frac{p^{n-1}_{i0} - p^{n-1}_{i,-1}}{h}\right)$$

$$- a\left(\theta D^2_x p^{n+1}_{i,-1/2} + (1-2\theta)D^2_x p^n_{i,-1/2} + \theta D^2_x p^{n-1}_{i,-1/2}\right) = 0$$

$$(5.20)\qquad p^0_{ij} = p^1_{ij} = 0 \qquad j = -\infty,\ldots,-1,$$

where

$$(5.21)\qquad p^n_{i,j-1/2} = 1/2\left(p^n_{ij} + p^n_{i,j-1}\right)$$

and

$$(5.22)\qquad D^2_x p^n_{ij} = \left(p^n_{i+1,j} - 2p^n_{ij} + p^n_{i-1,j}\right)/h^2,$$

is weakly stable in the sense that $\|p^n_h\| = \{\sum^0_{j=-\infty}\sum^\infty_{i=-\infty}(p^n_{ij})^2h^2\}^{1/2}$ may be bounded in terms of a constant dependent on n and on f and its derivatives up through second order.

We point out that for $\theta=1/2$ the numerical scheme given by (5.18)-(5.19) is equivalent to that described by (5.5),(5.10)-(5.14) for the boundary condition (5.15) or (5.16).

Proof :

Define $\mathcal{L}_h$ to be the discrete analogue of the operator $\mathcal{L}$ given by

$$\mathcal{L}(p) = \frac{\partial^2 p}{\partial t^2} + c\frac{\partial p}{\partial t \partial z} - a\frac{\partial^2 p}{\partial x^2}$$

for the above difference scheme :

$$\mathcal{L}_h(p^n_{ij}) = \frac{1}{\Delta t^2}\left(p^{n+1}_{i,j-1/2} - 2p^n_{i,j-1/2} + p^{n-1}_{i,j-1/2}\right) + \frac{c}{2\Delta t}\left(\frac{p^{n+1}_{ij} - p^{n+1}_{i,j-1}}{h} - \frac{p^{n-1}_{ij} - p^{n-1}_{i,j-1}}{h}\right)$$
$$- a\left(\theta D_x^2 p^{n+1}_{i,j-1/2} + (1-2\theta) D_x^2 p^n_{i,j-1/2} + \theta D_x^2 p^{n-1}_{i,j-1/2}\right) \tag{5.23}$$

Now if p^n_{ij} is the solution of (5.18)-(5.20) then $v^n_{i,j-1/2}$ given by

$$v^n_{i,j-1/2} = \mathcal{L}_h(p^n_{ij}) \qquad j = -\infty,\dots,0\ ;\ i = -\infty,\dots,\infty\ ;\ n = 2,\dots\infty$$
$$v^0_{i,j-1/2} = f^0_{i,j-1/2} - \frac{1}{2}\frac{c\Delta t}{h}(f^0_{ij} - f^0_{i,j-1}) - a\Delta t^2\,\theta D_x^2 f^0_{i,j-1/2} \tag{5.24}$$
$$v^1_{i,j-1/2} = f^1_{i,j-1/2} + \frac{1}{2}\frac{c\Delta t}{h}(f^1_{ij} - f^1_{i,j-1}) - a\Delta t^2\,\theta D_x^2 f^1_{i,j-1/2}$$

may also be defined as the solution of the Dirichlet problem

$$\Box_h v^n_{i,j-1/2} = \mathcal{L}_h(f^n_{ij}) \qquad i = -\infty,\dots,\infty\ ;\ j = -\infty,\dots,-1\ ;\ n = 2,\dots,\infty$$
$$v^n_{i,-1/2} = 0 \qquad i = -\infty,\dots,\infty\ ;\ n = 2,\dots,\infty$$
$$v^0_{i,j-1/2} = f^0_{i,j-1/2} - \frac{1}{2}\frac{c\Delta t}{h}(f^0_{ij} - f^0_{i,j-1}) - a\Delta t^2\,\theta D_x^2 f^0_{i,j-1/2} \tag{5.25}$$
$$v^1_{i,j-1/2} = f^1_{i,j-1/2} + \frac{1}{2}\frac{c\Delta t}{h}(f^1_{ij} - f^1_{i,j-1}) - a\Delta t^2\,\theta D_x^2 f^1_{i,j-1/2}$$

and we have the estimate

$$\| \bar{v}^n_h \| \le \text{const}\left(\| \bar{v}^0_h \| + \| \bar{v}^1_h \| + \| \bar{\mathcal{L}}_h f^n \|\right) \le V^n, \tag{5.26}$$

for some constant V^n depending on the derivatives of f up through second order, in the norm

$$\| q^n_h \| = \left\{ \sum_{j=-\infty}^{0} \sum_{i=-\infty}^{\infty} \left(q^n_{i,j-1/2}\right)^2 h^2 \right\}^{1/2}. \tag{5.27}$$

Then p^n_{ij} is the solution of

$$\mathcal{L}_h(p^n_{ij}) = v^n_{i,j-1/2} \qquad j = -\infty,\dots,0\ ;\ i = -\infty,\dots,\infty\ ;\ n = 2,3\dots$$
$$p^0_{ij} = 0 \qquad j = -\infty,\dots,0\ ;\ i = -\infty,\dots,\infty \tag{5.28}$$
$$p^1_{ij} = 0 \qquad j = -\infty,\dots,0\ ;\ i = -\infty,\dots,\infty$$

The idea of the proof, now following the development in [1] for the continuous paraxial approximation of the wave equation, is to estimate $\|p^n_h\|$ in terms of $\|v^n_h\|$. This could not be done if $\mathcal{L}_h$ were replaced by an operator $\tilde{\mathcal{L}}_h$ corresponding to the explicit scheme (5.6)-(5.9) in the definition of v^n_{ij}.

To bound $\| \overline{p}^{\,n}_h \|$, we multiply (5.28) by

$$\left(p^{n+1}_{i,j-1/2}-p^{n-1}_{i,j-1/2}\right)/(2\Delta t),$$

sum on i and multiply by h to obtain

$$\frac{h}{2\Delta t}\sum_i\left[\left(\frac{p^{n+1}_{i,j-1/2}-p^{n}_{i,j-1/2}}{\Delta t}\right)^2-\left(\frac{p^{n}_{i,j-1/2}-p^{n-1}_{i,j-1/2}}{\Delta t}\right)^2\right]$$

$$+\,c\sum_i\left[\left(\frac{p^{n+1}_{ij}-p^{n-1}_{ij}}{2\Delta t}\right)^2-\left(\frac{p^{n+1}_{i,j-1}-p^{n-1}_{i,j-1}}{2\Delta t}\right)^2\right]$$

$$-\,ah\sum_i D_x^2 p^n_{i,j-1/2}\left(\frac{p^{n+1}_{i,j-1/2}-p^{n-1}_{i,j-1/2}}{2\Delta t}\right)$$

$$-\,a\theta h\sum_i D_x^2\left(p^{n+1}_{i,j-1/2}-2p^{n}_{i,j-1/2}+p^{n-1}_{i,j-1/2}\right)\frac{p^{n+1}_{i,j-1/2}-p^{n-1}_{i,j-1/2}}{2\Delta t}$$

$$=h\sum_i v^n_{i,j-1/2}\left(\frac{p^{n+1}_{i,j-1/2}-p^{n-1}_{i,j-1/2}}{2\Delta t}\right).$$

We shall need some notation :

$$p^n_h=\left\{p^n_{ij}\right\}_{\substack{i=-\infty,\dots,\infty\\ j=-\infty,\dots,0}} \qquad \overline{p}^{\,n}_h=\left\{p^n_{i,j-1/2}\right\}_{\substack{i=-\infty,\dots,\infty\\ j=-\infty,\dots,0}}$$

$$p^n_{hj}=\left\{p^n_{ij}\right\}_{i=-\infty,\dots,\infty}$$

$$\left(p^n_{hj},q^m_{h\ell}\right)=h\sum_{i=-\infty}^{\infty}p^n_{ij}q^m_{i\ell} \qquad |p^n_{hj}|=\left(p^n_{hj},p^n_{hj}\right)^{1/2}$$

(5.29)

$$\left(\left(p^n_h,q^m_h\right)\right)=h\sum_{j=-\infty}^{0}\left(p^n_{hj},q^m_{hj}\right) \qquad \|p^n_h\|=\left(\left(p^n_h,p^n_h\right)\right)^{1/2}$$

$$d\left(p^n_{hj},q^m_{h\ell}\right)=h\sum_{i=-\infty}^{\infty}\left(\frac{p^n_{ij}-p^n_{i-1,j}}{h},\ \frac{q^m_{i\ell}-q^m_{i-1,\ell}}{h}\right)$$

$$d\left(\left(p^n_h,q^m_h\right)\right)=h\sum_{j=-\infty}^{0}d\left(p^n_{hj},q^m_{hj}\right).$$

Now we sum over j, j = -∞,0 and multiply by h. Standard calculations lead to

$$\frac{1}{2\Delta t}\left\{\left\| \frac{\bar{p}_h^{n+1}-\bar{p}_h^{n}}{\Delta t}\right\|^2 - \left\| \frac{\bar{p}_h^{n}-\bar{p}_h^{n-1}}{\Delta t}\right\|^2\right\} + c\left| \frac{p_{h0}^{n+1}-p_{h0}^{n-1}}{2\Delta t}\right|^2$$

$$+\frac{a}{2\Delta t}\left\{ d\left(\left(\frac{\bar{p}_h^{n+1}+\bar{p}_h^{n}}{2}, \frac{\bar{p}_h^{n+1}+\bar{p}_h^{n}}{2}\right)\right) - d\left(\left(\frac{\bar{p}_h^{n}+\bar{p}_h^{n-1}}{2}, \frac{\bar{p}_h^{n}+\bar{p}_h^{n-1}}{2}\right)\right)\right\}$$

$$+\frac{a\Delta t}{2}(\theta-1/4)\left\{ d\left(\left(\frac{\bar{p}_h^{n+1}-\bar{p}_h^{n}}{\Delta t}, \frac{\bar{p}_h^{n+1}-\bar{p}_h^{n}}{\Delta t}\right)\right) - d\left(\left(\frac{\bar{p}_h^{n}-\bar{p}_h^{n-1}}{\Delta t}, \frac{\bar{p}_h^{n}-\bar{p}_h^{n-1}}{\Delta t}\right)\right)\right\}$$

$$= \left(\left(\bar{v}_h^{n}, \frac{\bar{p}_h^{n+1}-\bar{p}_h^{n-1}}{2\Delta t}\right)\right).$$

Define the energy form :

$$E^{n+1/2} = \left\| \frac{\bar{p}_h^{n+1}-\bar{p}_h^{n}}{\Delta t}\right\|^2 + a\, d\left(\left(\frac{\bar{p}_h^{n+1}+\bar{p}_h^{n}}{2}, \frac{\bar{p}_h^{n+1}+\bar{p}_h^{n}}{2}\right)\right) + a\Delta t^2(\theta-1/4)\, d\left(\left(\frac{\bar{p}_h^{n+1}-\bar{p}_h^{n}}{\Delta t}, \frac{\bar{p}_h^{n+1}-\bar{p}_h^{n}}{\Delta t}\right)\right) \tag{5.30}$$

We have

$$\frac{1}{2\Delta t}\left(E^{n+1/2}-E^{n-1/2}\right) + c\left| \frac{p_{h0}^{n+1}-p_{h0}^{n-1}}{2\Delta t}\right|^2 = \left(\left(\bar{v}_h^{n}, \frac{\bar{p}_h^{n+1}-\bar{p}_h^{n-1}}{2\Delta t}\right)\right).$$

Summing over n, n=1,...,N and multiplying by $2\Delta t$ we obtain

$$E^{N+1/2} + 2c\sum_{n=1}^{N}\left| \frac{p_{h0}^{n+1}-p_{h0}^{n-1}}{2\Delta t}\right|^2 \Delta t = E^{1/2} + 2\sum_{n=1}^{N}\left(\left(\bar{v}_h^{n}, \frac{\bar{p}_h^{n+1}-\bar{p}_h^{n-1}}{2\Delta t}\right)\right)\Delta t.$$

If $\theta \geq 1/4$, we have

$$\left\| \frac{\bar{p}_h^{N+1}-\bar{p}_h^{N}}{\Delta t}\right\|^2 \leq E^{N+1/2} \leq 2\sum_{n=1}^{N}\left(\frac{\|\bar{v}_h^{n+1}\| + \|\bar{v}_h^{n}\|}{2}\right)\left\| \frac{\bar{p}_h^{n+1}-\bar{p}_h^{n}}{\Delta t}\right\| \Delta t.$$

Then applying a discrete Grönwall lemma we obtain

$$\left\| \frac{\bar{p}_h^{N+1}-\bar{p}_h^{N}}{\Delta t}\right\| \leq 2\sum_{n=1}^{N+1}\|\bar{v}_h^{n}\|\,\Delta t, \tag{5.31}$$

and

$$\|\bar{p}_h^{N+1}\| \leq 2N\Delta t\sum_{n=1}^{N+1}\|\bar{v}_h^{n}\|\,\Delta t. \tag{5.32}$$

Thus we have bounded $\|\bar{p}^N_h\|$ in terms of $t^N \sum^N_{n=1} \|v^n_h\|\,\Delta t$, where $\|\bar{p}^n_h\| = \sum_j\sum_i (\|p^n_{i,j-1/2}\|^2h^2)^{1/2}$. In order to obtain a bound for $\|p^n_h\| = \sum_j\sum_i \|p^n_{ij}\|^2h^2)^{1/2}$ it suffices to bound $\|D_z p^n_h\|$ where by $D_z p^n_h$ we mean

$$D_z p_h^n = \left\{\frac{p_{ij}^n-p_{i,j-1}^n}{h}\right\}_{\substack{i=-\infty,\dots,\infty\\ j=-\infty,\dots,0}} \tag{5.33}$$

Towards this end we return to (5.28) and multiply by the following discrete analogue of $(\partial p/\partial t)+c(\partial p/\partial z)$:

$$\frac{p^{n+1}_{i,j-1/2}-p^{n-1}_{i,j-1/2}}{2\Delta t}+c\left(\theta\,\frac{p^{n+1}_{ij}-p^{n+1}_{i,j-1}}{h}+(1-2\theta)\,\frac{p^{n}_{ij}-p^{n}_{i,j-1}}{h}+\theta\,\frac{p^{n-1}_{ij}-p^{n-1}_{i,j-1}}{h}\right).$$

After summing on i and j and multiplying by h^2 we obtain :

$$\frac{1}{2\Delta t}\left(\tilde{E}^{n+1/2}-\tilde{E}^{n-1/2}\right)+(\theta-1/4)\,\frac{c\Delta t^2}{2}\left|\frac{p^{n+1}_{h0}-2p^{n}_{h0}+p^{n-1}_{h0}}{\Delta t^2}\right|^2$$

$$+\;ac\,d\left(\theta p^{n+1}_{h0}+(1-2\theta)\,p^{n}_{h0}+\theta p^{n-1}_{h0},\;\theta p^{n+1}_{h0}+(1-2\theta)\,p^{n}_{h0}+\theta p^{n-1}_{h0}\right)$$

$$=\left(\left(v^n_h\;,\;\frac{\bar{p}^{n+1}_h-\bar{p}^{n-1}_h}{2\Delta t}+c\left(\theta\,\frac{p^{n+1}_{ij}-p^{n+1}_{i,j-1}}{h}+(1-2\theta)\,\frac{p^{n}_{ij}-p^{n}_{i,j-1}}{h}+\theta\,\frac{p^{n-1}_{ij}-p^{n-1}_{i,j-1}}{h}\right)\right)\right)$$

where the energy $\tilde{E}^{n+1/2}$ is given by :

$$\tilde{E}^{n+1/2}=\left\|\frac{\bar{p}^{n+1}_h-\bar{p}^{n}_h}{\Delta t}+c\left(\frac{D_zp^{n+1}_h+D_zp^{n}_h}{2}\right)\right\|^2+(\theta-1/4)\,c^2\,\|D_zp^{n+1}_h-D_zp^{n}_h\|^2$$

$$+\;a\,d\left(\left(\frac{\bar{p}^{n+1}_h+\bar{p}^{n}_h}{2},\frac{\bar{p}^{n+1}_h+\bar{p}^{n}_h}{2}\right)\right)+(\theta-1/4)\,a\Delta t^2\,d\left(\left(\frac{\bar{p}^{n+1}_h-\bar{p}^{n}_h}{\Delta t},\frac{\bar{p}^{n+1}_h-\bar{p}^{n}_h}{\Delta t}\right)\right)$$

Thus for $\theta\geq 1/4$ we have :

$$\left\|\frac{\bar{p}^{N+1}_h-\bar{p}^{N}_h}{\Delta t}+c\,\frac{D_zp^{N+1}_h+D_zp^{N}_h}{2}\right\|^2+(\theta-1/4)\,c^2\,\|D_zp^{N+1}_h-D_zp^{N}_h\|^2$$

$$\leq\; E^{N+1/2}$$

$$\leq\; E^{1/2}+2\sum_{n=1}^{N}\left(\left(\bar{v}^n_h\;,\;\frac{\bar{p}^{n+1}_h-\bar{p}^{n-1}_h}{2\Delta t}+\frac{1}{4}\,c\left(D_zp^{n+1}_h+2D_zp^{n}_h+D_zp^{n-1}_h\right)\right)\right)\Delta t$$

$$+\;2\sum_{n=1}^{N}\left(\left(\bar{v}^n_h\;,\;(\theta-1/4)\,c\left(D_zp^{n+1}_h-2D_zp^{n}_h+D_zp^{n-1}_h\right)\right)\right)\Delta t$$

$$\leq\; 2\sum_{n=1}^{N}\left(\frac{\|\bar{v}^n_h\|+\|\bar{v}^{n+1}_h\|}{2}\right)\left\|\frac{\bar{p}^{n+1}_h-\bar{p}^{n}_h}{\Delta t}+c\,\frac{D_zp^{n+1}_h+D_zp^{n}_h}{2}\right\|\Delta t$$

$$+\;2\sqrt{\theta-1/4}\;2\sum_{n=1}^{N}\left(\frac{\|\bar{v}^n_h\|+\|\bar{v}^{n+1}_h\|}{2}\right)\sqrt{\theta-1/4}\;\;c\,\|D_zp^{n+1}_h-D_zp^{n}_h\|\,\Delta t.$$

Again using a discrete Grönwall lemma we obtain

$$(5.35)\qquad \left\{ \left\| \frac{\bar{p}_h^{N+1} - \bar{p}_h^{N}}{\Delta t} + c\,\frac{D_z p_h^{N+1} + D_z p_h^{N}}{2} \right\|^2 + (\theta - 1/4)\, c^2 \, \| D_z p_h^{N+1} - D_z p_h^{N} \|^2 \right\}^{1/2}$$

$$\leq \; 2\,(1 + \sqrt{\theta - 1/4}\,) \sum_{n=1}^{N} \| \bar{v}_h^{n} \| \, \Delta t.$$

Now (5.31) together with (5.35) imply that for $1/4 < \theta \leq 1/2$:

$$\| D_z p_h^N \| \leq (3 + \sqrt{\theta - 1/4}\,) \sum_{n=1}^{N} \| \bar{v}_h^{n} \|,$$

and the theorem follows.

6 - Numerical results

In the numerical experiments presented here we compare the results obtained in Ω_F using an approximate impedance operator Z_a with those obtained by solving the transmission problem in $\Omega_F \cup \Omega_S$. We look at two cases. In both Ω_S is a homogeneous acoustic medium. In the first, the velocity C_S in Ω_S is 1/2 so that $Z_N \in \Sigma^S{}_a$ for all N and the initial boundary value problem in Ω_F is strongly well posed. (The velocity in Ω_F is C_F=1). In the second, C_S=5 so that for $N \geq 2$, Z_N is in $\Sigma^W{}_a$ but not in $\Sigma^S{}_a$ and the initial boundary value problem is only weakly well posed.

To obtain the reference solution, the so called exact solution, in $\Omega_F \cup \Omega_S$ we place the lower boundary as well as the lateral boundaries of the domain of calculation sufficiently far away from the source that they don't affect the numerical solution. (The upper boundary is the free boundary Γ_F). Then the simulation in Ω_F using an impedance operator Z_m is carried out by bringing the lower boundary up to Γ.

The source is a point source, the product of a Gaussian g(t) in time and a Dirac h(x,z) in space :

$$f(x,z,t) = h(x,z)g(t)$$

$$g(t) = \begin{cases} \exp(-10(1-t_S)^2) & 0 < t < 2t_S \\ 0 & 2t_S < t \end{cases}$$

$$h(x,z) = \begin{cases} 10^4(1-r/r_S) & r < r_S \\ 0 & r_S \leq r \end{cases}$$

$$r = ((x-x_S)^2 + (z-z_S)^2)^{1/2}$$

This source is centered at a point $(x_S - z_S) \in \Omega_F$ at a distance 0.3 from Γ and has a radius r_S=0.08. It is active between instants t=0 and $t=2t_S=0.2$.

The grid is a regular grid with $\Delta x = \Delta z = 0.01$ which corresponds to roughly 10 points per wave length effectively represented. The time step Δt is $\Delta t = \bar{c}\ h/2$ where $\bar{c}$ is the larger of the two velocities c_F and c_S.

Figures 4-7 correspond to the case c = 1/2. The surface shown represents the pressure p(x,z,t) in Ω_F at instant t=0.75. In addition to the exact solution we show the solutions obtained in Ω_F with the impedance operators Z_1, Z_2 and Z_3 given by (4.16), (4.17) and (4.18). The explicit numerical scheme (5.6)-(5.9) was used for the calculations on Γ. There is no problem of stability. All three impedance operators yield solutions which are excellant approximations to the exact solution.

Figures 8-14 correspond to the case c=5. Again the pressure p(x,z,t) in Ω_F at t=0.75 is shown. The exact solution is given in *Figure 8* and that obtained with the impedance operator $Z_1 \in \Sigma^S{}_a$, (4.16), is shown in *Figure 9*. We see that stability considerations aside, the approximation of the true impedance is more difficult here than in the preceeding case. Besides the exact solution, there are two solutions obtained with the impedance operator Z_2 given by (4.17), one shown in *Figure 10* with the explicit numerical scheme on Γ, (5.6)-(5.9), and the other *Figure 11* with the implicit scheme on Γ, (5.10)-(5.13). In *Figure 10* we see typical indications of instability, and the energy curve exploded for this simulation.

In *Figure 11* the solution, while more reasonable than that of *Figure 10* , is not entirely precise. There is a larger reflection for larger angles of incidence than is correct and this reflection does propagate into the domain Ω_F. The energy shown in *Figure 13* increases linearly in time. These are both characteristics of weak stability. Whether this provides a better approximation of the measured solution than the solution obtained with an impedance operator in $\Sigma^S{}_a$ is likely to depend on the zone of observation or measure, cf. Section 7.

Figure 12 is obtained with an operator Z_a corresponding to $r_a(\theta)$ given by (4.21) for ε=0. In this case $r_a(1) = 1$ and while Z_a is not in $\Sigma^S{}_a$ there are no indications of instability ; the energy, *Figure 14* , decreases (cf.Remark 4.1).

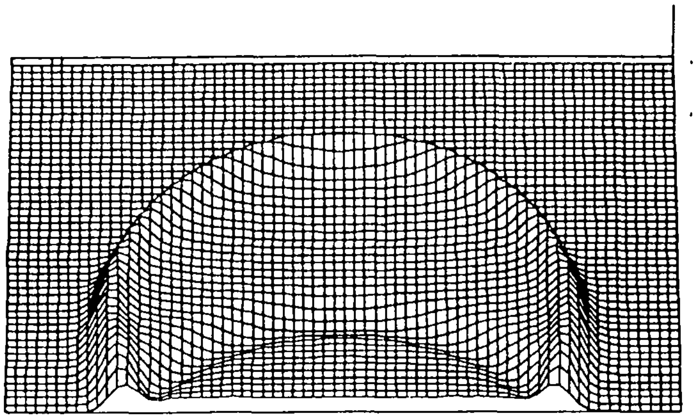


Fig. 4 Solution at t = 0.75
♦ cas c = 1/2
♦ exact solution
min = 0.02 , max = 1.43

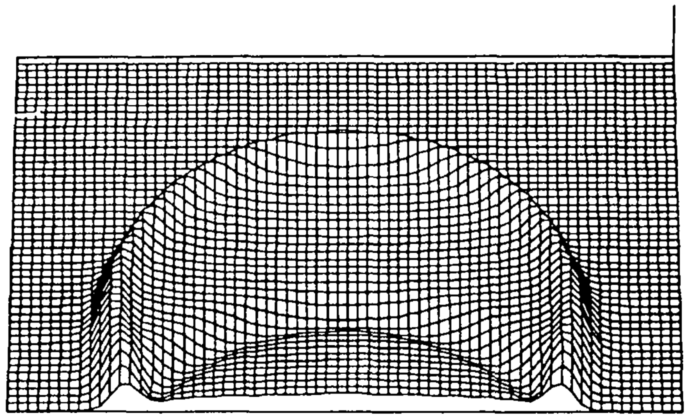


Fig. 5 Solution at t = 0.75
♦ cas c = 1/2
♦ solution with boundary condition of order 1
min = 0.04 , max = 1.43

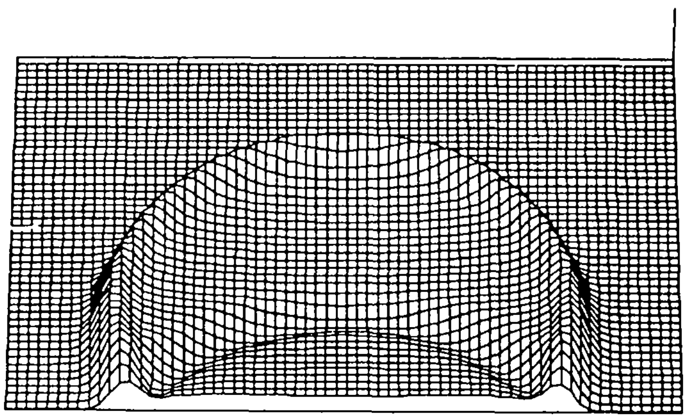


Fig. 6 Solution at t = 0.75
♦ cas c = 1/2
♦ solution with boundary condition of order 2
min = 0.02 , max = 1.43

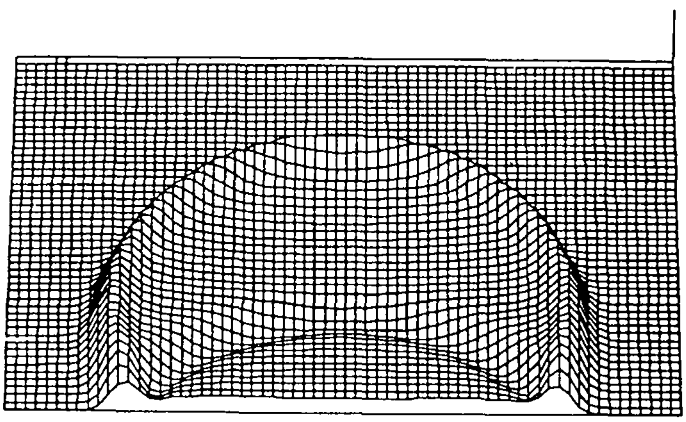


Fig. 7 Solution at t = 0.75
♦ cas c = 1/2
♦ solution with boundary condition of order 3
min = 0.02 , max = 1.43

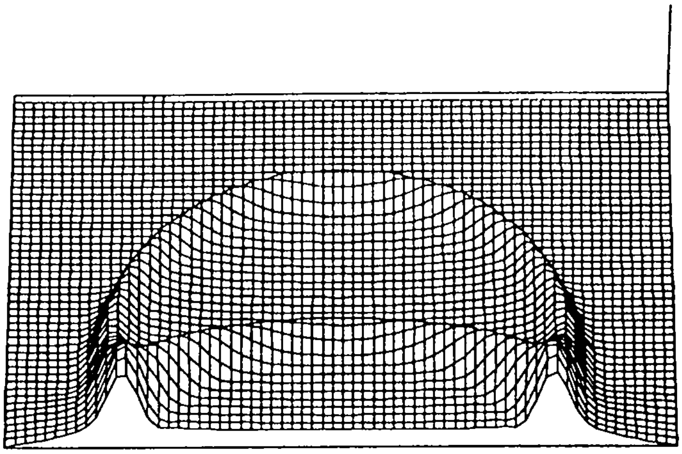


Fig. 8 Solution at t = 0.75
♦ cas c = 5
♦ exact solution
min = 0. , max = 2.48

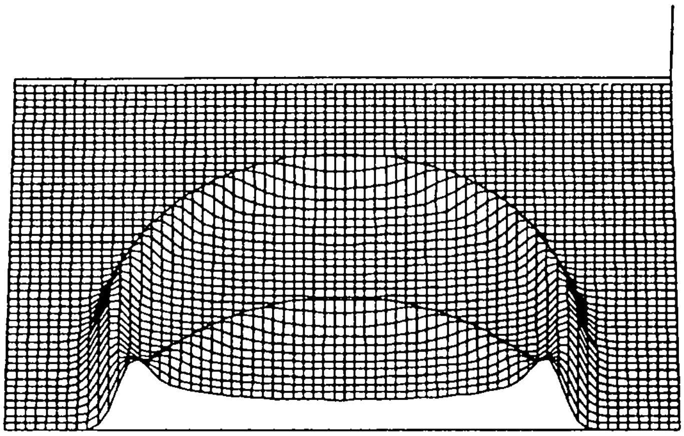


Fig. 9 Solution at t = 0.75
♦ cas c = 5
♦ solution with boundary condition of order 1
min = 0. , max = 2.02

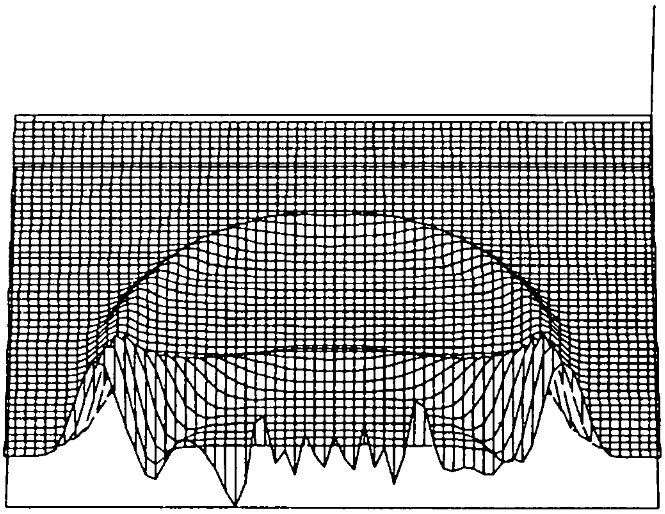


Fig. 10 Solution at t = 0.75
♦ cas c = 5
♦ solution with boundary condition of order 2
♦ explicit numerical scheme
min = -3.03 , max = 6.28

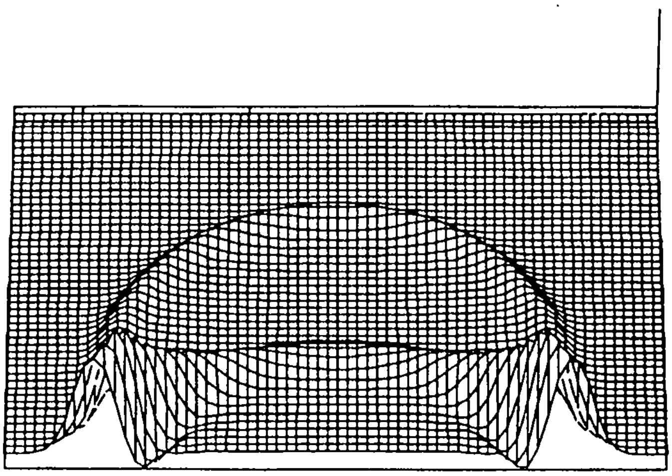


Fig. 11 Solution at t = 0.75
♦ cas c = 5
♦ solution with boundary condition of order 2
♦ implicit numerical scheme
min = 0.02 , max = 1.43

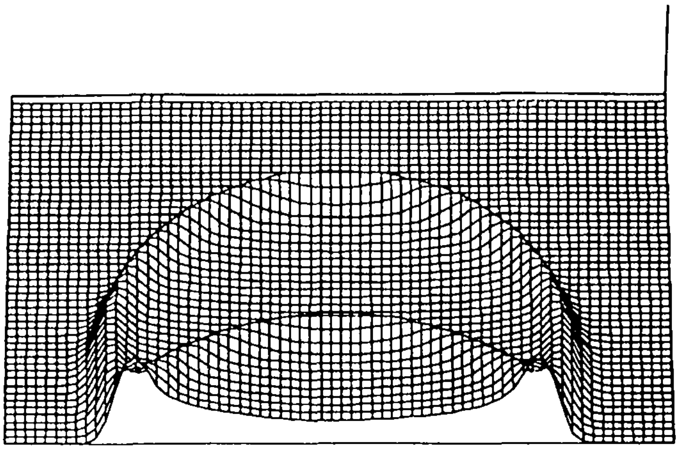


Fig. 12 Solution at t = 0.75

- cas c = 5
- solution with boundary condition associated with (4.21) $\varepsilon = 0$
- explicit numerical scheme

min = 0. , max = 2.49

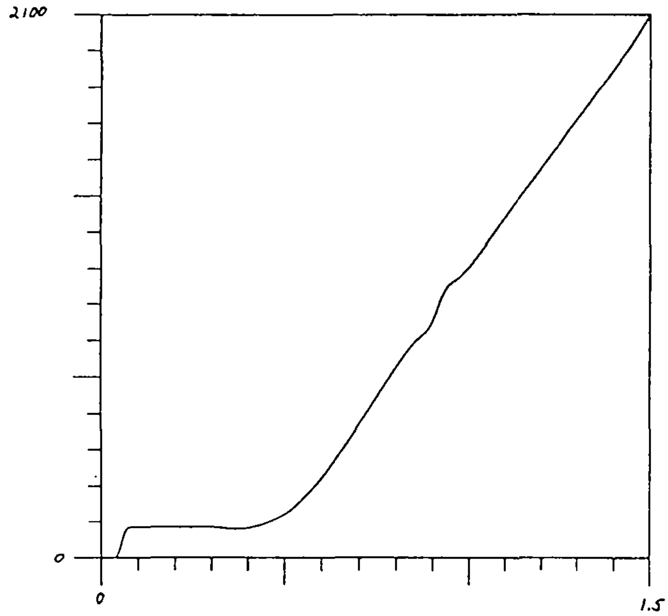


Fig. 13 Energy as a function of time

- cas c = 5
- boundary condition of order 1
- implicit numerical scheme

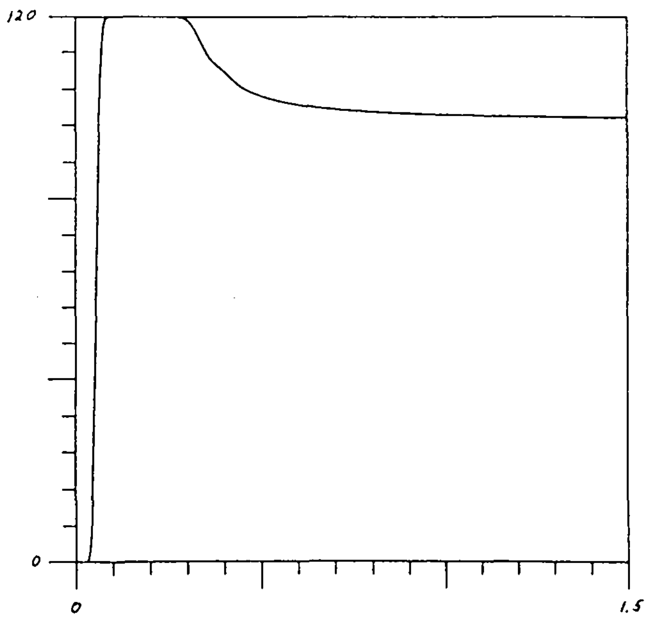


Fig. 14 Energy as a function of time

- cas c = 5
- boundary condition associated with (4.21), $\varepsilon = 0$
- explicit numerical scheme

7 - Conclusion : perspectives for the inverse problem

Problem (5.4) defines a mapping F associating to $Z \in \Sigma^S{}_a$ the solution p of (5.4). For a subset $\mathcal{O}$ of Ω_F, which we shall call the *domain of observation,* we denote by $F_{\mathcal{O}}$, the mapping F followed by restriction to $\mathcal{O}$: $F_{\mathcal{O}}(Z) = p|_{\mathcal{O}}$. The question which arises is can we determine Z from an observation of the solution p in $\mathcal{O}$; can we invert $F_{\mathcal{O}}$. More precisely, suppose that by an experiment we have observed the exact solution $p_{\mathcal{O}}(x,z,t)$ corresponding to the true impedance operator $Z \in \Sigma$ in the set $\mathcal{O}$ during the time interval [0,T]. Can we find the approximate impedance operator $Z_a \in \Sigma^N{}_a$ such that $F_{\mathcal{O}}(Z)$ best approximates, in the sense of least squares for example, the observation $p_{\mathcal{O}}$, where by $\Sigma^N{}_a$ we denote the subclass of $\Sigma^w{}_a$ consisting of those impedance operators whose complex impedance is of the form (3.4) for some $N' \leq N$. To find such a Σ_a it suffices to solve the following minimization problem. Find $Z^* \in \Sigma^W{}_a$ such that :

$$J(Z^*) \leq J(Z) \text{ for all } Z \in \Sigma_a^N$$

(7.1)

$$J(Z) = \left(\int_0^T \| F_{\mathcal{O}}(Z) - p_{\mathcal{O}} \|^2_{L^2(\mathcal{O})} dt \right)^{1/2}$$

Problem (7.1) is a problem of optimal control which we call the *inverse problem* . The function J is called the *criterion* or *cost function* . Problem (5.4) is called the *direct problem* or *equation of state* .

To solve (7.1) we need a minimization algorithm. All algorithms for the minimization of differentiable functions are iterative methods which make use of the gradient of J. To calculate J we solve a system of equations called the *adjoint equation* . This system is of the same type as is the state equation (5.4) and all the well posedness properties given in section 4 apply as well to the adjoint equation and we use the same numerical scheme to solve it.

Thus we have defined a space of approximate impedance operators which satisfies for the most part the desired properties (1)-(4) of section 3. Each element of $\Sigma^N{}_a$ is determined by a small number of parameters, at least for N small. Each element of $\Sigma^N{}_a$ leads to a well posed direct problem as well as a to a well posed adjoint equation. We have defined a numerical scheme to solve the direct problem that is stable and simple to solve and which should work equally well for the adjoint equation.

For the inverse problem we can of course envisage several variations on the model problem (7.1) such as repeating the experiment with different source terms, or introducing a weight in the norm used to define the optimization criterion J. However we insist on pointing out the difficulty of the preposed inverse problem. The function F_Θ being very non linear, the fonctional we have to minimize is anything but quadratic. Consequently the problem of existence and especially uniqueness of the solution of the inverse problem is entirely open. Furthermore, the functional J being non convex we can expect the existence of local minima which renders even more arduous the task of the optimization algorithm.

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